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K-MAP MINIMIZATION

A Karnaugh map provides a systematic method for simplifying Boolean expressions and, if properly used, will produce the simplest SOP or POS expression possible, known as the minimum expression. As you have seen, the effectiveness of algebraic simplification depends on your familiarity with all the laws, rules, and theorems of Boolean algebra and on your ability to apply them. The Karnaugh map, on the other hand, provides a "cookbook" method for simplification.

A Karnaugh map is similar to a truth table because it presents all of the possible values of input variables and the resulting output for each value. Instead of being organized into columns and rows like a truth table, the Karnaugh map is an array of cells in which each cell represents a binary value of the input variables. The cells are arranged in a way so that simplification of a given expression is simply a matter of properly grouping the cells. Karnaugh maps can be used for expressions with two, three, four, and five variables. Another method, called the Quine-McClusky method can be used for higher numbers of variables.

The number of cells in a Karnaugh map is equal to the total number of possible input variable combinations as is the number of rows in a truth table. For three variables, the number of cells is $2^3 = 8$. For four variables, the number of cells is $2^4 = 16$.

The 3-Variable Karnaugh Map

The 3-variable Karnaugh map is an array of eight cells, as shown in Fig.(1)(a). In this case, A, B, and C are used for the variables although other letters could be used. Binary values of A and B are along the left side (notice the sequence) and the values of C are across the top. The value of a given cell is the binary values of A and B at the left in the same row combined with the value of C at the top in the same column. For example, the cell in the upper left corner has a binary value of 000 and the cell in the lower right corner has a binary value of 101. Fig.(1)(b) shows the standard product terms that are represented by each cell in the Karnaugh map.

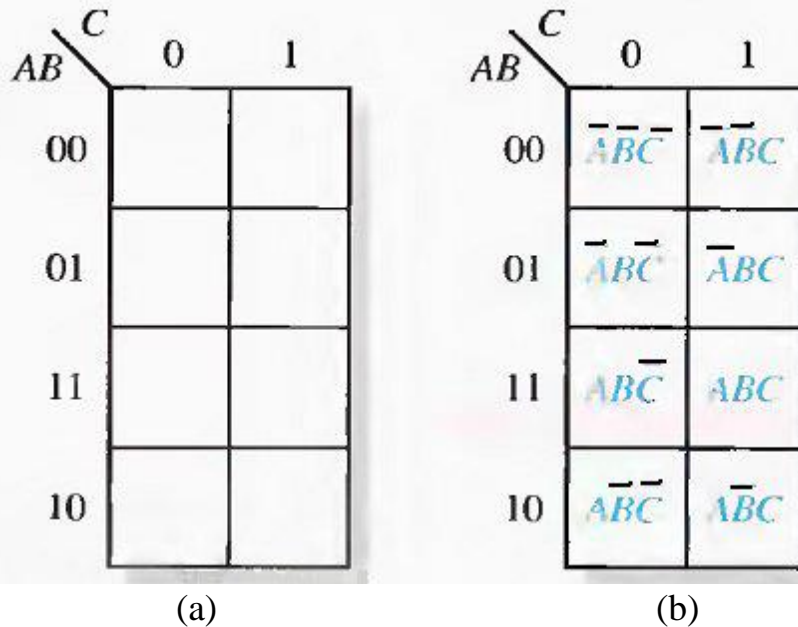


Fig.(1) A 3-variable Karnaugh map showing product terms.

The 4-Variable Karnaugh Map

The 4-variable Karnaugh map is an array of sixteen cells, as shown in Fig.(2)(a). Binary values of A and B are along the left side and the values of C and D are across the top. The value of a given cell is the binary values of A and B at the left in the same row combined with the binary values of C and D at the top in the same column. For example, the cell in the upper right corner has a binary value of 0010 and the cell in the lower right corner has a binary value of 1010. Fig.(2)(b) shows the standard product terms that are represented by each cell in the 4-variable Karnaugh map.

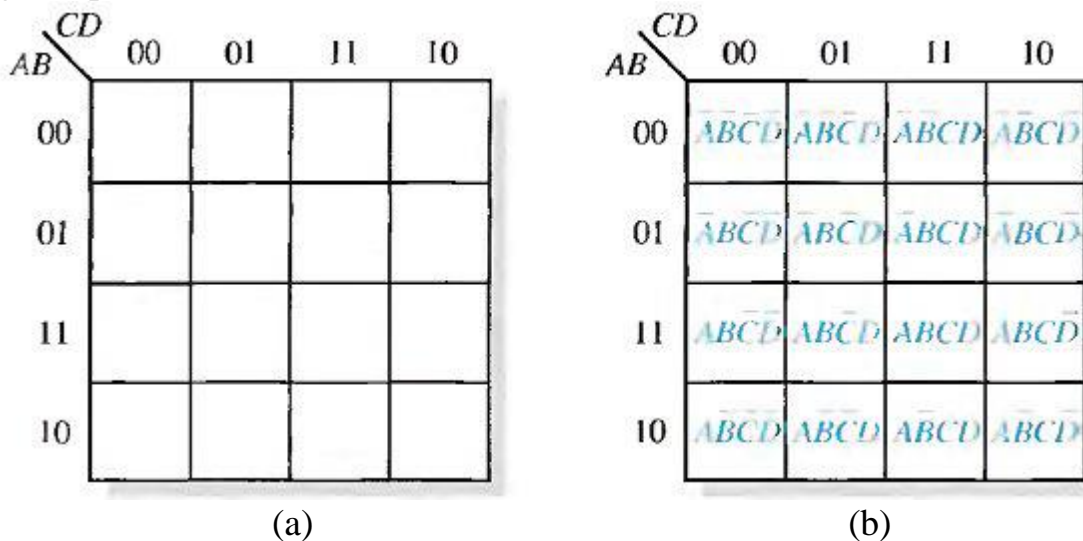


Fig.(2) A 4-variable Karnaugh map.

Cell Adjacency

The cells in a Karnaugh map are arranged so that there is only a single-variable change between adjacent cells. Adjacency is defined by a single variable change. In the 3-variable map the 010 cell is adjacent to the 000 cell, the 011 cell, and the 110 cell. The 010 cell is not adjacent to the 001 cell, the 111 cell, the 100 cell, or the 101 cell.

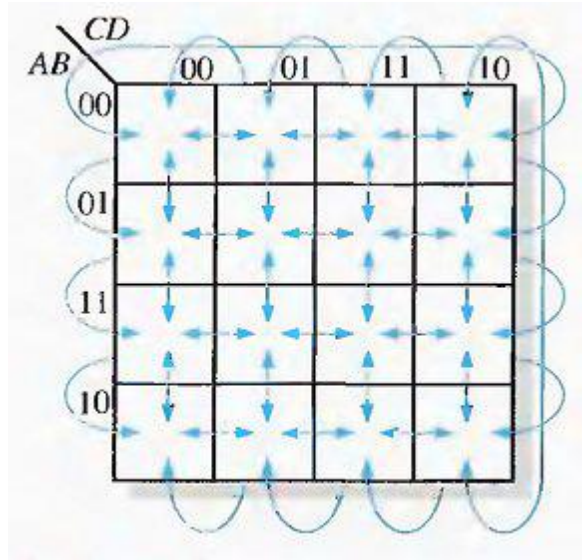


Fig.(3) Adjacent cells on a Karnaugh map are those that differ by only one variable. Arrows point between adjacent cells.

KARNAUGH MAP SOP MINIMIZATION

For an SOP expression in standard form, a 1 is placed on the Karnaugh map for each product term in the expression. Each 1 is placed in a cell corresponding to the value of a product term. For example, for the product term ABC, a 1 goes in the 101 cell on a 3-variable map.

Example

Map the following standard SOP expression on a Karnaugh map:
see Fig.(4).

Example

Map the following standard SOP expression on a Karnaugh map:

$$\overline{A}BCD + \overline{A}BC\overline{D} + A\overline{B}CD + ABCD + ABC\overline{D} + \overline{A}\overline{B}CD + \overline{A}BC\overline{D}$$

See Fig.(5).

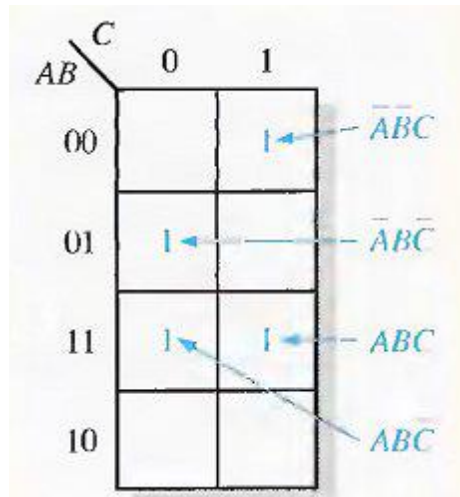


Fig.(4)

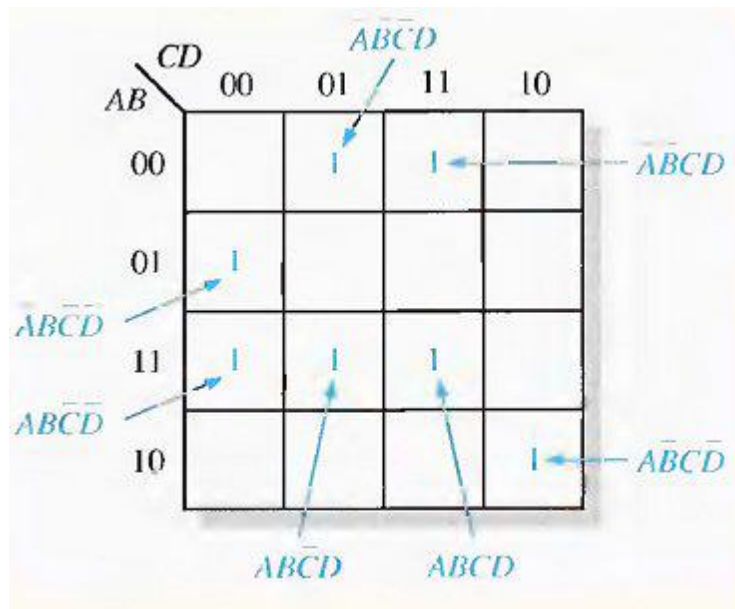


Fig.(5)

Example

Map the following SOP expression on a Karnaugh map:

$$\bar{A} + \bar{A}\bar{B} + \bar{A}B\bar{C}$$

Solution

The SOP expression is obviously not in standard form because each product term does not have three variables. The first term is missing two variables, the second term is missing one variable, and the third term is standard. First expand the terms numerically as follows:

$$\bar{A} + \bar{A}\bar{B} + \bar{A}\bar{B}\bar{C}$$

000	100	110
001	101	
010		
011		

		C	
		0	1
AB	00	1	1
	01	1	1
	11	1	
	10	1	1

Example

$$\bar{B}\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}CD$$

Solution

The SOP expression is obviously not in standard form because each product term does not have four variables.

$\bar{B}\bar{C}$	$\bar{A}\bar{B}$	+	$\bar{A}\bar{B}\bar{C}$	+	$\bar{A}\bar{B}\bar{C}\bar{D}$	+	$\bar{A}\bar{B}\bar{C}D$	+	$\bar{A}\bar{B}CD$
0000	1000		1100		1010		0001		1011
0001	1001		1101						
1000	1010								
1001	1011								

Map each of the resulting binary values by placing a 1 in the appropriate cell of the 4- variable Karnaugh map.

		CD			
		00	01	11	10
AB	00	1	1		
	01				
	11	1	1		
	10	1	1	1	1

Karnaugh Map Simplification of SOP Expressions

Grouping the 1s, you can group 1s on the Karnaugh map according to the following rules by enclosing those adjacent cells containing 1s. The goal is to maximize the size of the groups and to minimize the number of groups.

- A group must contain either 1, 2, 4, 8, or 16 cells, which are all powers of two. In the case of a 3-variable map, $2^3 = 8$ cells is the maximum group.
- Each cell in a group must be adjacent to one or more cells in that same group.
- Always include the largest possible number of 1s in a group in accordance with rule 1.
- Each 1 on the map must be included in at least one group. The 1s already in a group can be included in another group as long as the overlapping groups include noncommon 1s.

Example:

Group the 1s in each of the Karnaugh maps in Fig.(6).

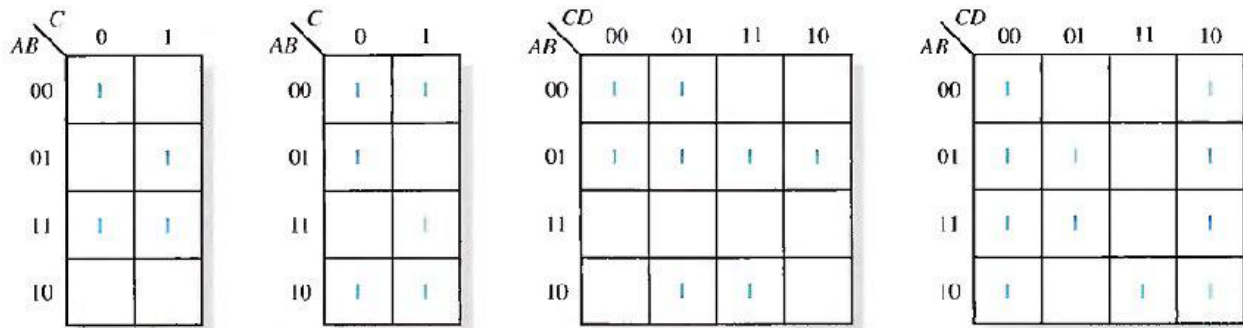


Fig.(6)

Solution:

The groupings are shown in Fig.(7). In some cases, there may be more than one way to group the 1s to form maximum groupings.

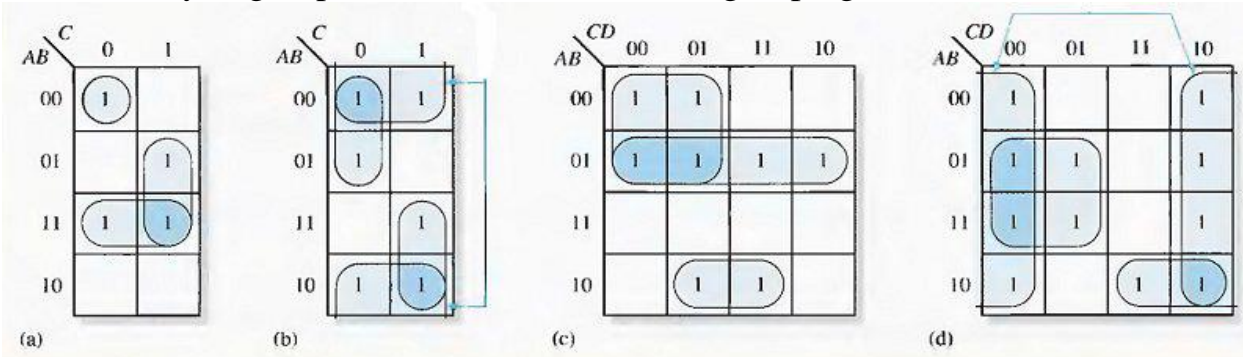


Fig.(7)

"Don't Care" Conditions

Sometimes a situation arises in which some input variable combinations are not allowed. For example, recall that in the BCD code there are six invalid combinations: 1010, 1011, 1100, 1101, 1110, and 1111. Since these unallowed states will never occur in an application involving the BCD code, they can be treated as "don't care" terms with respect to their effect on the output. That is, for these "don't care" terms either a 1 or a 0 may be assigned to the output: it really does not matter since they will never occur.

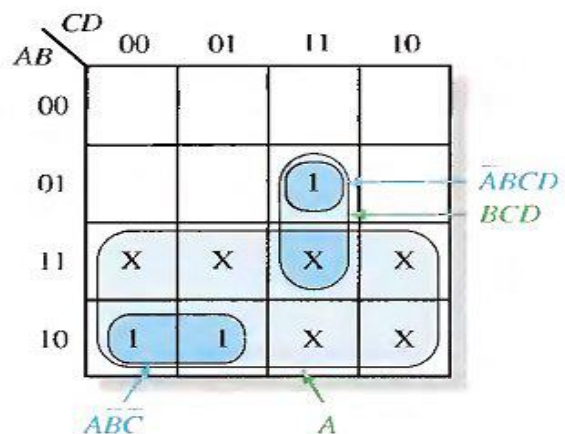
The "don't care" terms can be used to advantage on the Karnaugh map. Fig.(9) shows that for each "don't care" term, an X is placed in the cell. When grouping the 1s, the Xs can be treated as 1s to make a larger grouping or as 0s if they cannot be used to advantage. The larger a group, the simpler the resulting term will be.

The truth table in Fig.(9)(a) describes a logic function that has a 1 output only when the BCD code for 7,8, or 9 is present on the inputs. If the "don't cares" are used as 1s, the resulting expression for the function is $A + BCD$, as indicated in part (b). If the "don't cares" are not used as 1s, the resulting expression is $ABC + ABCD$: so you can see the advantage of using **"don't care"** terms to get the simplest expression.

Inputs				Output
A	B	C	D	Y
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	1
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

(a) Truth table

Don't cares



(b) Without "don't cares" $Y = ABC + ABCD$
With "don't cares" $Y = A + BCD$

Fig.(9)

KARNAUGH MAP POS MINIMIZATION

In this section, we will focus on POS expressions. The approaches are much the same except that with POS expressions, 0s representing the standard sum terms are placed on the Karnaugh map instead of 1s.

For a POS expression in standard form, a 0 is placed on the Karnaugh map for each sum term in the expression. Each 0 is placed in a cell corresponding to the value of a sum term. For example, for the sum term $A + \bar{B} + C$, a 0 goes in the 0 1 0 cell on a 3-variable map.

When a POS expression is completely mapped, there will be a number of 0s on the Karnaugh map equal to the number of sum terms in the standard POS expression. The cells that do not have a 0 are the cells for which the expression is 1. Usually, when working with POS expressions, the 1s are left off. The following steps and the illustration in Fig.(10) show the mapping process.

Step 1. Determine the binary value of each sum term in the standard POS expression. This is the binary value that makes the term equal to 0.

Step 2. As each sum term is evaluated, place a 0 on the Karnaugh map in the corresponding cell.

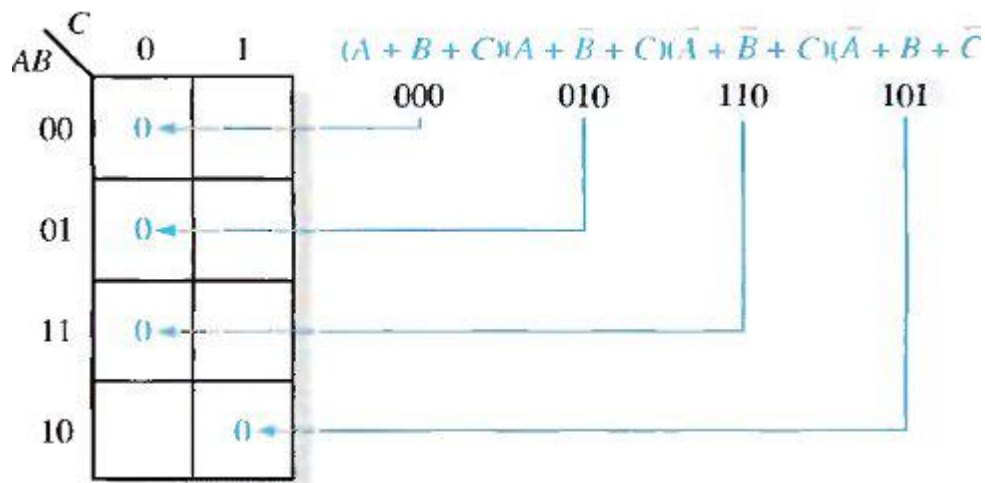


Fig.(10)

Example of mapping a standard POS expression.

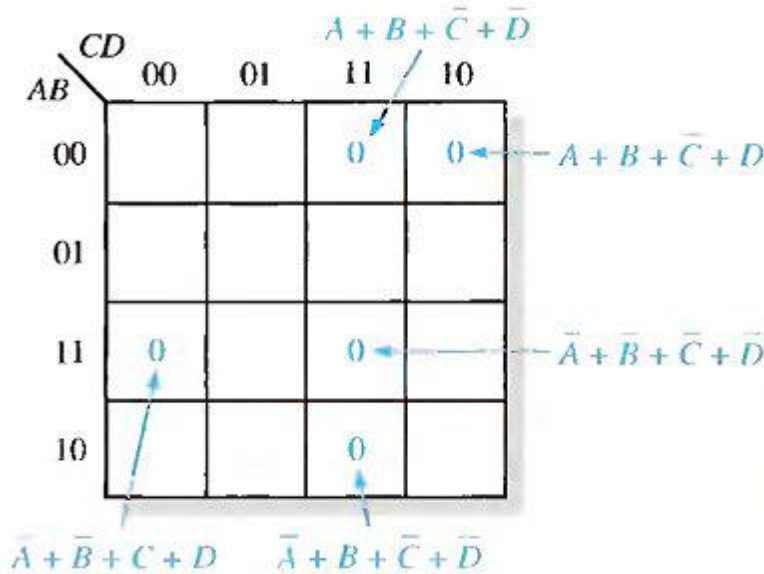
Example:

Map the following standard POS expression on a Karnaugh map:

$$(\bar{A} + \bar{B} + C + D)(\bar{A} + B + \bar{C} + \bar{D})(A + B + \bar{C} + D)(\bar{A} + \bar{B} + \bar{C} + \bar{D})(A + B + \bar{C} + \bar{D})$$

Solution:

$$\begin{array}{cccccc}
 (\bar{A} + \bar{B} + C + D) & (\bar{A} + B + \bar{C} + \bar{D}) & (A + B + \bar{C} + D) & (\bar{A} + \bar{B} + \bar{C} + \bar{D}) & (A + B + \bar{C} + \bar{D}) \\
 1100 & 1011 & 0010 & 1111 & 0011
 \end{array}$$



Karnaugh Map Simplification of POS Expressions

The process for minimizing a POS expression is basically the same as for an SOP expression except that you group 0s to produce minimum sum terms instead of grouping 1s to produce minimum product terms. The rules for grouping the 0s are the same as those for grouping the 1s that you learned before.

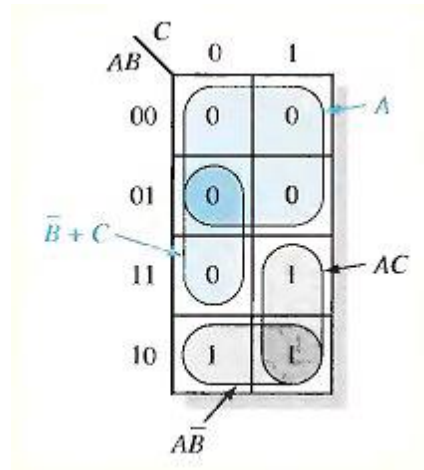
Example:

Use a Karnaugh map to minimize the following standard POS expression:

Also, derive the equivalent SOP expression.

$$(A + B + C)(A + B + \bar{C})(A + \bar{B} + C)(A + \bar{B} + \bar{C})(\bar{A} + \bar{B} + C)$$

Solution:



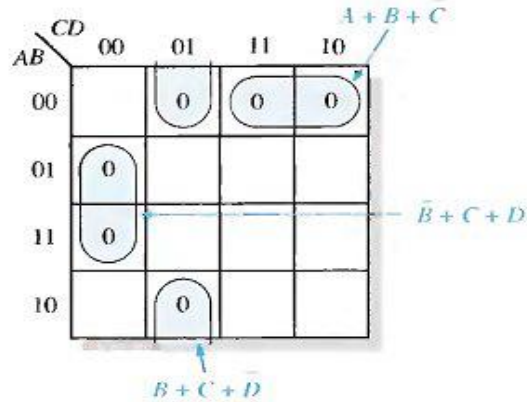
Example: Use a Karnaugh map to minimize the following POS expression:

$$(B + C + D)(A + B + \bar{C} + D)(\bar{A} + B + C + \bar{D})(A + \bar{B} + C + D)(\bar{A} + \bar{B} + C + D)$$

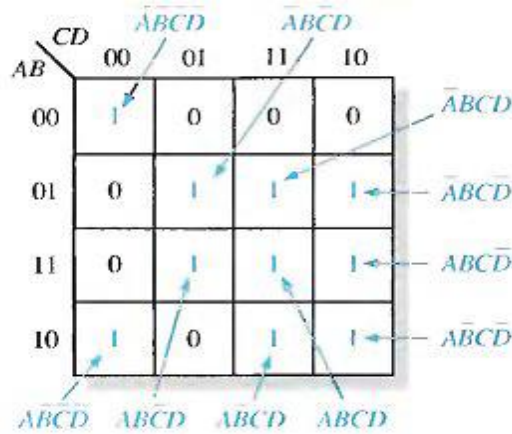
Example: Using a Karnaugh map, convert the following standard POS expression into a minimum POS expression, a standard SOP expression, and a minimum SOP expression.

$$(\bar{A} + \bar{B} + C + D)(A + \bar{B} + C + D)(A + B + C + \bar{D})$$

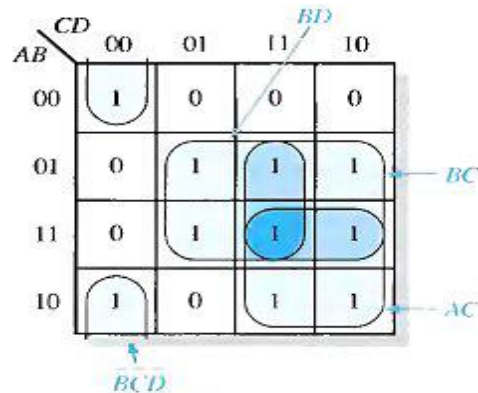
$$(A + B + \bar{C} + \bar{D})(\bar{A} + B + C + \bar{D})(A + B + \bar{C} + D)$$



(a) Minimum POS: $(A + B + C)(\bar{B} + \bar{C} + D)(B + C + \bar{D})$



(b) Standard SOP:
 $\bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}CD + \bar{A}B\bar{C}\bar{D} + \bar{A}BC\bar{D} + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D} +$
 $A\bar{B}CD + ABC\bar{D} + ABCD + ABCD$



(c) Minimum SOP: $AC + BC + BD + \bar{B}\bar{C}\bar{D}$

FIVE-VARIABLE KARNAUGH MAPS

Boolean functions with five variables can be simplified using a 32-cell Karnaugh map. Actually, two 4-variable maps (16 cells each) are used to construct a 5-variable map. You already know the cell adjacencies within each of the 4-variable maps and how to form groups of cells containing 1s to simplify an SOP expression. All you need to learn for five variables is the cell adjacencies between the two 4-variable maps and how to group those adjacent 1s.

A Karnaugh map for five variables (ABCDE) can be constructed using two 4-variable maps with which you are already familiar. Each map contains 16 cells with all combinations of variables B, C, D, and E. One map is for $A = 0$ and the other is for $A = 1$, as shown in Fig.(11).

Cell Adjacencies

You already know how to determine adjacent cells within the 4-variable map. The best way to visualize cell adjacencies between the two 16-cell maps is to imagine that the $A = 0$ map is placed on top of the $A = 1$ map. Each cell in the $A = 1$ map is adjacent to the cell directly below it in the $A = 0$ map, see Fig.(12).

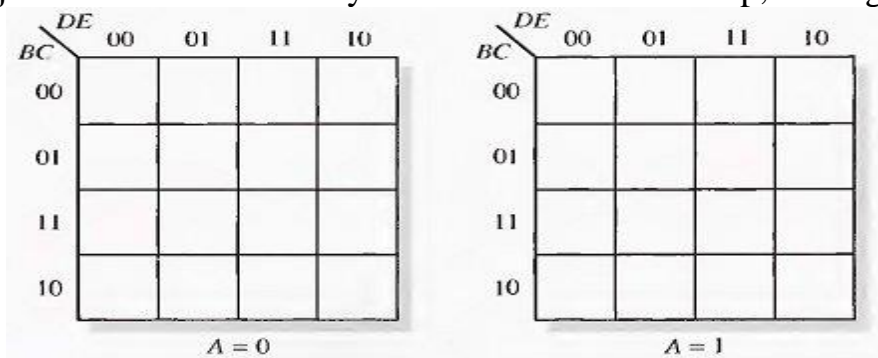


Fig.(11)

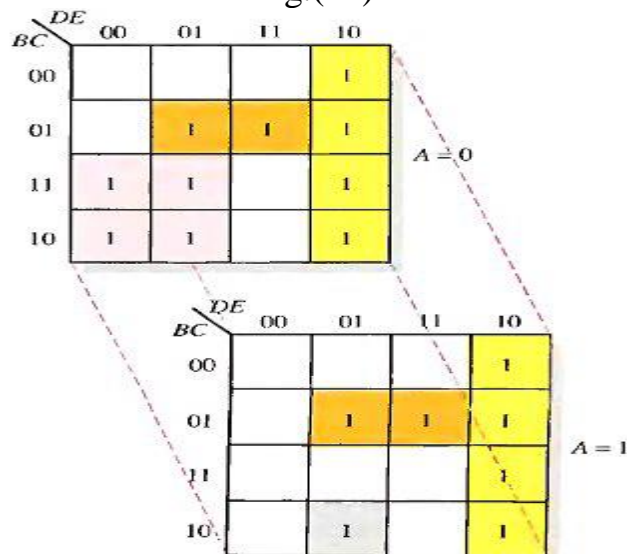


Fig.(12)

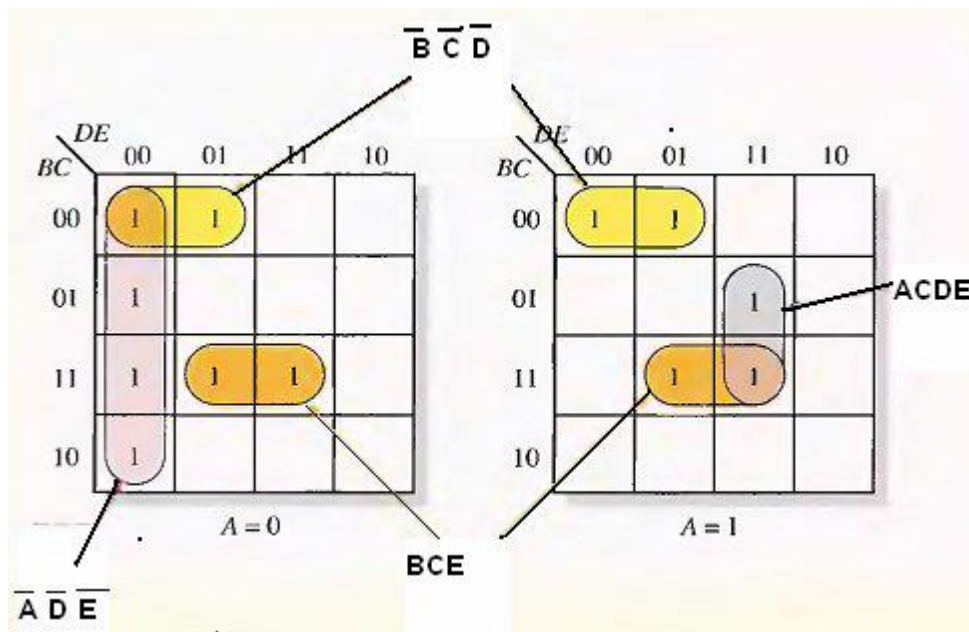
The simplified SOP expression yields

$$x = D\bar{E} + \bar{B}CE + \bar{A}\bar{B}\bar{D} + B\bar{C}\bar{D}E$$

Example:

Use a Karnaugh map to minimize the following standard SOP 5-variable expression:

$$X = \bar{A}\bar{B}\bar{C}\bar{D}\bar{E} + \bar{A}\bar{B}\bar{C}\bar{D}E + \bar{A}\bar{B}\bar{C}D\bar{E} + \bar{A}\bar{B}\bar{C}DE + \bar{A}\bar{B}C\bar{D}\bar{E} + \bar{A}\bar{B}C\bar{D}E + \bar{A}\bar{B}CD\bar{E} + \bar{A}\bar{B}CDE + \bar{A}BC\bar{D}\bar{E} + \bar{A}BC\bar{D}E + \bar{A}BCD\bar{E} + \bar{A}BCDE + A\bar{B}\bar{C}\bar{D}\bar{E} + A\bar{B}\bar{C}\bar{D}E + A\bar{B}\bar{C}D\bar{E} + A\bar{B}\bar{C}DE + A\bar{B}C\bar{D}\bar{E} + A\bar{B}C\bar{D}E + A\bar{B}CD\bar{E} + A\bar{B}CDE + AB\bar{C}\bar{D}\bar{E} + AB\bar{C}\bar{D}E + AB\bar{C}D\bar{E} + AB\bar{C}DE + ABC\bar{D}\bar{E} + ABC\bar{D}E + ABCD\bar{E} + ABCDE$$



$$X = \bar{A}D\bar{E} + \bar{B}\bar{C}\bar{D} + BCE + ACDE$$