**Plane Electromagnetic Waves in a Conducting Medium**

Let us consider a linear, homogeneous and isotropic conducting medium characterised by constant permittivity $\varepsilon$, permeability $\mu$ and conductivity $\sigma$. To simplify the discussion we assume that the medium is charge free ($\rho = 0$) and external-current free such that the currents existing in the medium are induced only by the electromagnetic wave itself. Thus, we take

$$\vec{J} = \sigma \vec{E}$$

Any initial charge distribution within the conductor dies out quickly. The charges move to the surface and make $\vec{J} = 0$ inside.

For such a medium Maxwell’s equations take the following form:

\[
\begin{align*}
\nabla \cdot \vec{E} &= 0 \quad (1a) \\
\nabla \cdot \vec{H} &= 0 \quad (1b) \\
\n\nabla \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \quad (1c) \\
\n\nabla \times \vec{H} &= \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad (1d)
\end{align*}
\]

Taking curl of Eq. (1c) and using Eq. (1d) we get

\[
\nabla \left( \nabla \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\mu \frac{\partial}{\partial t} \left( \nabla \times \vec{H} \right) = -\sigma \mu \frac{\partial \vec{E}}{\partial t} - \varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}. \quad (2a)
\]
Now using Eq. (1a) in Eq. (2a) we get
\[
\nabla^2 \vec{E} - \sigma \mu \frac{\partial \vec{E}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0. \tag{3a}
\]

Similarly, taking curl of Eq. (1d) and using Eqs. (1c) and (1a) we can get
\[
\nabla^2 \vec{H} - \sigma \mu \frac{\partial \vec{H}}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0. \tag{3b}
\]

Now, the plane wave solutions are
\[
\vec{E} (\vec{r}, t) = \vec{E}_0 e^{i(k \cdot \vec{r} - \omega t)} \tag{4a}
\]
\[
\vec{H} (\vec{r}, t) = \vec{H}_0 e^{i(k \cdot \vec{r} - \omega t)} \tag{4b}
\]

where $\vec{E}_0$ and $\vec{H}_0$ are complex amplitudes which are constants in space and time,

$k = k \hat{n}$ is the propagation vector

Substituting the solution (4a) in Eq. (3a) and (4b) in Eq. (3b) we get
\[
[-k^2 + j \omega \sigma \mu + \omega^2 \epsilon \mu] \vec{E} = 0.
\]

For nonzero solution,
\[
k^2 = \omega^2 \epsilon \mu + j \omega \sigma \mu. \tag{5a}
\]

Let
\[k = \alpha + j \beta\]

Then we get
Comparing Eqs. (5a) and (5b) we get

\[ k^2 = \alpha^2 - \beta^2 + j2\alpha\beta. \]  

(5b)

On solving equations (6a) and (6b) we get

\[ \alpha^2 - \beta^2 = \omega^2 \varepsilon \mu \]  

(6a)

\[ 2\alpha\beta = \omega \sigma \mu. \]  

(6b)

\[ \alpha = \omega \sqrt{\frac{\varepsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} + 1 \right]^{1/2}, \]  

(7a)

\[ \beta = \omega \sqrt{\frac{\varepsilon \mu}{2}} \left[ \sqrt{1 + \left( \frac{\sigma}{\omega \varepsilon} \right)^2} - 1 \right]^{1/2}. \]  

(7b)

In terms of \( \alpha \) and \( \beta \) the field vectors become

\[ \vec{E}(\vec{r}, t) = \vec{E}_0 \cdot e^{-\beta \hat{n} \cdot \vec{r}} \cdot e^{j(\alpha \hat{n} \cdot \vec{r} - \omega t)} \]  

(8a)

\[ \vec{H}(\vec{r}, t) = \vec{H}_0 \cdot e^{-\beta \hat{n} \cdot \vec{r}} \cdot e^{j(\alpha \hat{n} \cdot \vec{r} - \omega t)}. \]  

(8b)

These solutions show that the field amplitudes are spatially attenuated.

- The physical reason of the attenuation is that the wave sets up electric current in the medium, which causes dissipation of energy in the form of Joule heating.
The quantity $\beta$, the imaginary part of wave number $\kappa$, is a measure of attenuation and is called attenuation constant. It depends on the frequency $\omega$ and conductivity $\sigma$.

For nonconductors $\sigma = 0$ and then we have $\beta = 0$ meaning that there is no attenuation of the field vectors.

For good conductors

$$\sigma / \omega \epsilon \gg 1$$

Then we have

$$\alpha \approx \beta \approx \sqrt{\frac{\omega \sigma \mu}{2}}. \quad (9a)$$

The quantity $1/\beta$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $1/e$ of its initial amplitude at the surface. It is known as skin depth or penetration depth into a conducting medium.

Thus, skin depth $\delta$ for a good conductor is

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega \sigma \mu}}. \quad (10)$$

- Skin depth $\delta$ decreases with increase in frequency and conductivity.

- For good conductors at high frequencies $\delta$ is very small. That is why in high frequency circuits current flows only through the surface of good conductors. This phenomenon is called skin effect. Due to this effect the ac resistance of a conductor is greater than its dc resistance. For this in high frequency circuits it is better to use a number of fine stranded wires instead of a thick wire. It increases surface area for a given area of cross-section and reduces resistance.
At microwave frequencies $\delta$ for Ag is very small ($\sim 10^{-3}$ mm). As a result, in the microwave region the performance of a waveguide made of pure Ag and another waveguide made of Ag-coated brass would appear to be identical. This technique is used to reduce the material cost of good conductors.

Substituting the solution (8a) in Eq. (1a) and solution (8b) in Eq. (1b) we get:

\begin{align*}
\vec{k} \cdot \vec{E} &= 0 \\
\vec{k} \cdot \vec{H} &= 0
\end{align*} \tag{11a, 11b}

These equations indicate that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation. So electromagnetic waves in a conducting medium are transverse in nature.

Again, the substitution of the solutions (8a) and (8b) in Eqs. (1c) and (1d) gives

\begin{align*}
jk \times \vec{E} &= -\mu \left( -j\omega \vec{H} \right) \quad \text{or} \quad k \times \vec{E} = \mu \omega \vec{H} \\
jk \times \vec{H} &= \sigma \vec{E} - j\omega \epsilon \vec{E} \quad \text{or} \quad k \times \vec{H} = - \left( \omega \epsilon + j\sigma \right) \vec{E}.
\end{align*} \tag{12a, 12b}

These two equations imply that $\vec{E}$ and $\vec{H}$ are mutually perpendicular and also they are perpendicular to the direction of propagation vector $\vec{k}$. 