Subject: Physics

Course- PG

Faculty- Science

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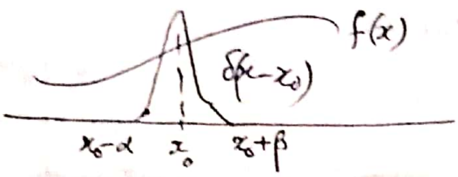
Properties of Dirac Delta Function

1. The integral of product of a function with Dirac delta function over all space is equal to the value of the function at the centre of Dirac delta function.

Let $\delta(x-x_0)$ be Dirac delta function which is centred at x_0. Let $f(x)$ be a continuous function of x. Then

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

The limits of integration may shrink to the region in which delta function is non-zero.



In the figure, for $\alpha=\beta$ tending to zero,

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = \int_{x_0-\alpha}^{x_0+\beta} f(x)\,\delta(x-x_0) = f(x_0)$$

2. Dirac Delta function is an even function. This means

$$\delta(x) = \delta(-x)$$

3. The product of x and $\delta(x)$ is zero.

4. The product of a function with delta function is equal to the product of the value of function at the centre of Dirac delta function and the Dirac delta function.

$$f(x)\,\delta(x) = f(0)\,\delta(x), \text{ or } f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$

5. Scaling: For real a,

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

6. $$\int \delta(x-a)\,\delta(x-b)\,dx = \delta(a-b)$$

7. $$\delta(x^2-a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|a|}$$

Derivative : The expression depends upon function used to define $\delta(x)$.

Let $\delta(x) = \lim_{L\to\infty} \left(\frac{\sin xL}{\pi x}\right)$

Then, $\frac{d}{dx}\delta(x) = \lim_{L\to\infty} \frac{d}{dx}\left(\frac{\sin xL}{\pi x}\right)$

$$= \lim_{L\to\infty}\left[\frac{L\cos Lx}{\pi x} - \frac{\sin xL}{\pi x^2}\right].$$

It can be shown for all representations that

$$x\frac{d}{dx}\delta(x) = -\delta(x).$$

The derivative is an odd function of x.

Transforms

1. Laplace Transform

$$L\{\delta(x-a)\} = \int_0^\infty e^{-sx}\,\delta(x-a)\,dx$$

$$= e^{-sa}$$

where we have used

$$\int_{\text{all space}} f(x)\,\delta(x-a)\,dx = f(a).$$

a. For $a=0$,

$$L\{\delta(x)\} = e^{-s.0}$$

$$= 1.$$

2. Fourier Transform

$$F\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\,\delta(x-a)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}e^{-isa}$$

For $a=0$,

$$F\{\delta(x)\} = \frac{1}{\sqrt{2\pi}}$$

Dirac Delta Function in 3D

It is defined as $\delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$ where $x\hat{i}+y\hat{j}+z\hat{k} = \vec{r}$. The integral condition is

$$\iiint \delta(\vec{r}-\vec{r}_0)\, d^3\vec{r} = 1.$$

Here $d^3\vec{r} = dx\,dy\,dz$, in cartesian System.

In spherical system

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3}\iiint e^{i\vec{k}\cdot\vec{r}}\, d^3\vec{k} \qquad \text{--- (1)}$$

Also

$$\iiint \frac{\delta(r)}{2\pi r^2}\, d^3\vec{r} = 1, \quad \text{as } d^3\vec{r} = r^2 \sin\theta\, d\theta\, d\varphi\, dr$$

$$\therefore\ \delta(\vec{r}) = \frac{\delta(|\vec{r}|)}{2\pi|\vec{r}|^2}. \qquad \text{--- (2)}$$

$$\delta(\vec{r}-\vec{r}_0) = \frac{\delta(r-r_0)}{2\pi r^2} \qquad \text{--- (3)}$$

In cylindrical system of co-ordinates

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

Hence

$$\iiint f(r,\theta,z)\,\delta(\vec{r}-\vec{r}_0)\, r\,dr\,d\theta\,dz = f(r_0,\theta_0,z_0)$$

which suggests that

$$\delta(\vec{r}-\vec{r}_0) = \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(z-z_0)}{r} \qquad \text{--- (4)}$$

S.A.

1. Define Dirac Delta function. Compare with Kronecker delta.
2. Write an explain any four relations as properties of Dirac Delta Function.

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